4. Affine transformations

Reading

Required:

- ◆ Watt, Section 1.1.

Further reading:

- ◆ Foley, et al, Chapter 5.1-5.5.
- ◆ David F. Rogers and J. Alan Adams, *Mathematical Elements for Computer Graphics*, 2nd Ed., McGraw-Hill, New York, 1990, Chapter 2.

Logistics:
- HW #1 handed out today.

- Project #1 due on Thursday, artifact on Monday.

- Sign up for grading times for Project #1.

Geometric transformations

Geometric transformations will map points in one space to points in another: $(x',y',z') = f(x,y,z)$.

These tranformations can be very simple, such as scaling each coordinate, or complex, such as non-linear twists and bends.

We'll focus on transformations that can be represented easily with matrix operations.

We'll start in 2D...

Representation

We can represent a **point**, **p** = (x,y), in the plane

- as a column vector

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

- as a row vector

$$\begin{bmatrix} x & y \end{bmatrix}$$

Representation, cont.

We can represent a **2-D transformation** *M* by a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If **p** is a column vector, *M* goes on the left:

$$\mathbf{p'} = M\mathbf{p}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

If **p** is a row vector, M^T goes on the right:

$$\mathbf{p'} = \mathbf{p}M^T$$

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

We will use **column vectors**.

Two-dimensional transformations

Here's all you get with a 2 x 2 transformation matrix *M*:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

So:

$$x' = ax + by$$
$$y' = cx + dy$$

We will develop some intimacy with the elements *a, b, c, d…*

Identity

Suppose we choose *a=d=1, b=c=0:*

- ◆ Gives the **identity** matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- ◆ Doesn't move the points at all

Scaling

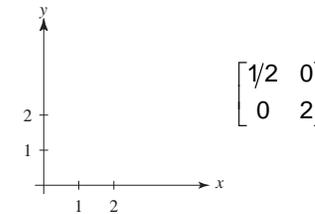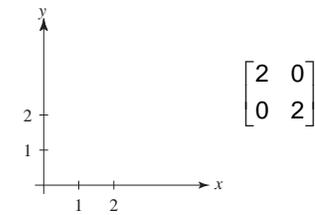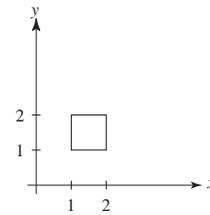Suppose we set *b=c=0*, but let *a* and *d* take on any *positive* value:

- ◆ Gives a **scaling** matrix:

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

- ◆ Provides **differential (non-uniform) scaling** in *x* and *y*:
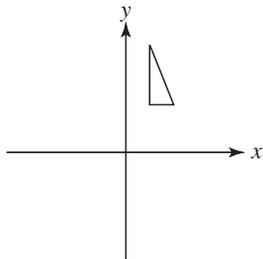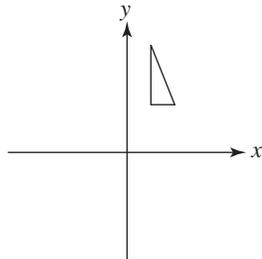
$$x' = ax$$
$$y' = dy$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$$

Suppose we keep *b=c=0*, but let either *a* or *d* go negative.

Examples:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

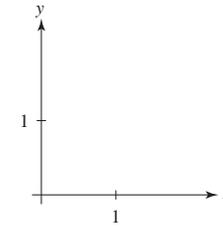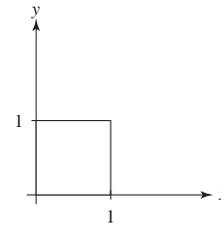$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Now let's leave *a=d=1* and experiment *b*. . . .

The matrix

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$
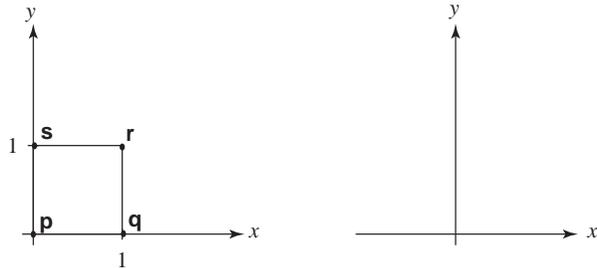
gives:

$$x' = x + by$$
$$y' = y$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

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Effect on unit square

Let's see how a general 2 x 2 transformation M affects the unit square:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \mathbf{p} & \mathbf{q} & \mathbf{r} & \mathbf{s} \end{bmatrix} = \begin{bmatrix} \mathbf{p'} & \mathbf{q'} & \mathbf{r'} & \mathbf{s'} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{bmatrix}$$
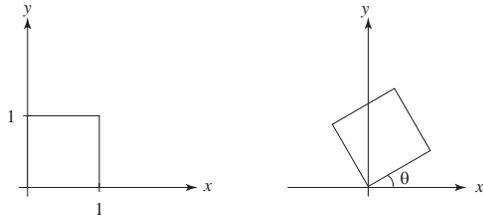
Effect on unit square, cont.

Observe:

- ◆ Origin invariant under M
- ◆ M can be determined just by knowing how the corners (1,0) and (0,1) are mapped
- ◆ a and d give x- and y-scaling
- ◆ b and c give x- and y-shearing

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Rotation

From our observations of the effect on the unit square, it should be easy to write down a matrix for "rotation about the origin":

- $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow$

- $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow$

Thus,

$$M = R(\theta) = \begin{bmatrix} & \\ & \end{bmatrix}$$

Linear transformations

The unit square observations also tell us the 2x2 matrix transformation implies that we are representing a point in a new coordinate system:

$$\mathbf{p}' = M\mathbf{p}$$
$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$= x \cdot \mathbf{u} + y \cdot \mathbf{v}$$

where $\mathbf{u} = [a\ c]^T$ and $\mathbf{v} = [b\ d]^T$ are vectors that define a new **basis** for a **linear space**.

The transformation to this new basis (a.k.a., change of basis) is a **linear transformation**.

Limitations of the 2 x 2 matrix

A 2 x 2 linear transformation matrix allows

- ◆ Scaling
- ◆ Rotation
- ◆ Reflection
- ◆ Shearing

Q: What important operation does that leave out?

Affine transformations

In order to incorporate the idea that both the basis and the origin can change, we augment the linear space **u**, **v** with an origin **t**.

We call **u**, **v**, and **t** (basis and origin) a **frame** for an **affine space**.

Then, we can represent a change of frame as:

$$\mathbf{p'} = x \cdot \mathbf{u} + y \cdot \mathbf{v} + \mathbf{t}$$

This change of frame is also known as an **affine transformation**.

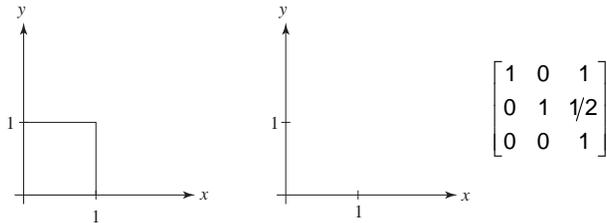How do we write an affine transformation with matrices?

Homogeneous coordinates

Idea is to loft the problem up into 3-space, adding a third component to every point:

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

And then transform with a 3 x 3 matrix:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = T(\mathbf{t}) \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$
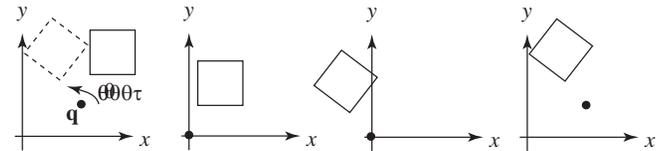
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

. . . gives **translation**!

Rotation about arbitrary points

Until now, we have only considered rotation about the origin.

With homogeneous coordinates, you can specify a rotation, θ, about any point $\mathbf{q} = [q_x \ q_y]^T$ with a matrix:

1. Translate $\mathbf{q}$ to origin

2. Rotate

3. Translate back

Note: Transformation order is important!!

Points and vectors

From now on, we can represent points as have an additional coordinate of $w=1$.

Vectors have an additional coordinate of $w=0$. Thus, a change of origin has no effect on vectors.

Q: What happens if we multiply a matrix by a vector?

These representations reflect some of the rules of affine operations on points and vectors:

$$\text{vector} + \text{vector} \rightarrow \text{vector}$$
$$\text{scalar} \cdot \text{vector} \rightarrow \text{vector}$$
$$\text{point} - \text{point} \rightarrow \text{vector}$$
$$\text{point} + \text{vector} \rightarrow \text{point}$$
$$\text{point} + \text{point} \rightarrow$$

One useful combination of affine operations is:

$$\mathbf{p}(t) = \mathbf{p}_0 + t\mathbf{u}$$

Q: What does this describe?

Barycentric coordinates

A set of points can be used to create an affine frame. Consider a triangle *ABC* and a point *P*:
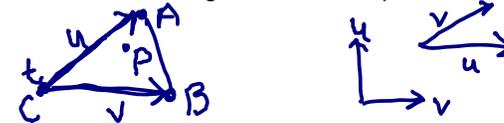
We can form a frame with an origin *C* and the vectors from *C* to the other vertices:

$$\mathbf{u} = A - C \quad \mathbf{v} = B - C \quad \mathbf{t} = C$$

We can then write *P* in this coordinate frame:

$$P = \alpha\mathbf{u} + \beta\mathbf{v} + \mathbf{t}$$
$$= \alpha(A - C) + \beta(B - C) + C$$
$$= \alpha A + \beta B + (1 - \alpha - \beta) C$$
$$= \alpha A + \beta B + \gamma C$$

The coordinates (α, β, γ) are called the **barycentric coordinates** of *P* relative to *A*, *B*, and *C*.

Computing barycentric coordinates

In the triangle example:

we can compute the barycentric coordinates of P:

$$\alpha A + \beta B + \gamma C = \begin{bmatrix} A_x & B_x & C_x \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} P_x \\ P_y \\ 1 \end{bmatrix}$$

Simple matrix analysis gives the solution:

$$\alpha = \frac{\begin{Vmatrix} P_x & B_x & C_x \\ P_y & B_y & C_y \\ 1 & 1 & 1 \end{Vmatrix}}{\begin{Vmatrix} A_x & B_x & C_x \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{Vmatrix}} \quad \beta = \frac{\begin{Vmatrix} A_x & P_x & C_x \\ A_y & P_y & C_y \\ 1 & 1 & 1 \end{Vmatrix}}{\begin{Vmatrix} A_x & B_x & C_x \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{Vmatrix}} \quad \gamma = \frac{\begin{Vmatrix} A_x & B_x & P_x \\ A_y & B_y & P_y \\ 1 & 1 & 1 \end{Vmatrix}}{\begin{Vmatrix} A_x & B_x & C_x \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{Vmatrix}}$$

Computing the determinant of the denominator gives:

$$B_x C_y - B_y C_x + A_y C_x - A_x C_y + A_x B_y - A_y B_x$$

Cross products

Consider the cross-product of two vectors, **u** and **v**. What is the geometric interpretation of this cross-product?

u×v
u × v
magnitude = this area

A cross-product can be computed as:

$$\mathbf{u} \times \mathbf{v} = \begin{Vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{Vmatrix}$$

$$= (u_y v_z - u_z v_y)\mathbf{i} + (u_z v_x - u_x v_z)\mathbf{j} + (u_x v_y - u_y v_x)\mathbf{k}$$

$$= \begin{bmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{bmatrix}$$

What happens when **u** and **v** lie in the *x-y* plane? What is the area of the triangle they span?

½ of magnitude z component

Barycentric coords from area ratios

Now, let's rearrange the equation from two slides ago:

$$B_x C_y - B_y C_x + A_y C_x - A_x C_y + A_x B_y - A_y B_x =$$
$$(B_x - A_x)(C_y - A_y) - (B_y - A_y)(C_x - A_x)$$

The determinant is then just the *z*-component of (B-A) x (C-A), which is two times the area of triangle *ABC*!

Thus, we find:

$$\alpha = \frac{\text{SArea}(PBC)}{\text{SArea}(ABC)} \quad \beta = \frac{\text{SArea}(APC)}{\text{SArea}(ABC)} \quad \gamma = \frac{\text{SArea}(ABP)}{\text{SArea}(ABC)}$$

Where SArea(RST) is the signed area of a triangle, which can be computed with cross-products.

Affine and convex combinations

Note that we seem to have added points together, which we said was illegal, but as long as they have coefficients that sum to one, it's ok.

We call this an **affine combination**. More generally:

$$P = \alpha_1 A_1 + \cdots + \alpha_n A_n$$

is a proper affine combination if:

$$\sum_{i=1}^{n} \alpha_i = 1$$

Note that if the α_i's are all positive, the result is more specifically called a **convex combination**.
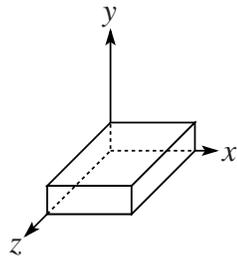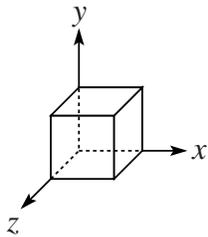
Q: Why is it called a convex combination?

Basic 3-D transformations: scaling

Some of the 3-D transformations are just like the 2-D ones.
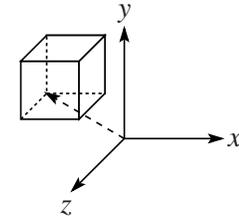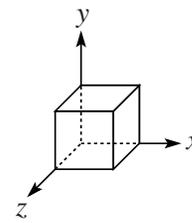
For example, scaling:

$$
\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
$$

Translation in 3D

Rotation...

← Transl

$$
\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
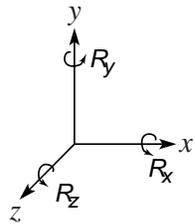$$

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Rotation in 3D

Rotation now has more possibilities in 3D:

Rotate @ x

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
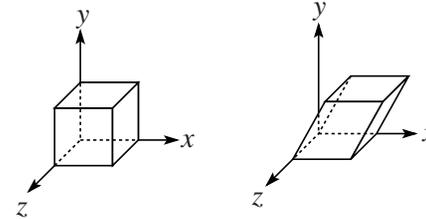
Use right hand rule

$R_1 R_2 R_3 \neq R_3 R_1 R_2$

Shearing in 3D

Shearing is also more complicated. Here is one example:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

We call this a shear with respect to the x-z plane.

Preservation of affine combinations

A transformation F is an affine transformation if it preserves affine combinations:

$$F(\alpha_1 A_1 + \cdots + \alpha_n A_n) = \alpha_1 F(A_1) + \cdots + \alpha_n F(A_n)$$

where the A_i are points, and:

$$\sum_{i=1}^{n} \alpha_i = 1$$

Clearly, the matrix form of F has this property.

One special example is a matrix that drops a dimension. For example:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$
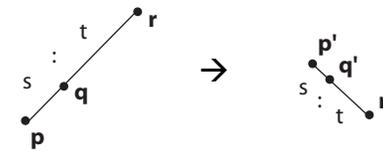
This transformation, known as an orthographic projection is an affine transformation.

We'll use this fact later…

Properties of affine transformations

Here are some useful properties of affine transformations:

- Lines map to lines
- Parallel lines remain parallel
- Midpoints map to midpoints (in fact, ratios are always preserved)

$$\text{ratio} = \frac{\|\mathbf{pq}\|}{\|\mathbf{qr}\|} = \frac{s}{t} = \frac{\|\mathbf{p'q'}\|}{\|\mathbf{q'r'}\|}$$

Summary

What to take away from this lecture:

- All the names in boldface.
- How points and transformations are represented.
- What all the elements of a 2 x 2 transformation matrix do and how these generalize to 3 x 3 transformations.
- What homogeneous coordinates are and how they work for affine transformations.
- How to concatenate transformations.
- The rules for combining points and vectors
- The mathematical properties of affine transformations.

Next class: Shading

Topics we'll cover:

- How does light interact with surfaces?

- What approximations do we use to model this interaction in computer graphics?

Read:
    Watt, sections 6.2 – 6.3

Optional Reading:
    Watt, chapter 7.